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New supersymmetric AdS_4 type II vacua

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ABSTRACT: Building on our recent results on dynamic $SU(3) \times SU(3)$ structures we present a set of sufficient conditions for supersymmetric $AdS_4 \times_w \mathcal{M}_6$ backgrounds of type IIA/IIB supergravity. These conditions ensure that the background solves, besides the supersymmetry equations, all the equations of motion of type II supergravity. The conditions state that the internal manifold is locally a codimension-one foliation such that the five dimensional leaves admit a Sasaki-Einstein structure. In type IIA the supersymmetry is $\mathcal{N} = 2$, and the six-dimensional internal space is locally an S^2 bundle over a four-dimensional Kähler-Einstein base; in IIB the internal space is the direct product of a circle and a five-dimensional squashed Sasaki-Einstein manifold. Given any five-dimensional Sasaki-Einstein manifold, we construct the corresponding families of type IIA/IIB vacua. The precise profiles of all the fields are determined at the solution and depend on whether one is in IIA or in IIB. In particular the background does not contain any sources, all fluxes (including the Romans mass in IIA) are generally non-zero, and the dilaton and warp factor are non-constant.

Contents

1. Introduction

In the absence of sources and higher-order derivative corrections, supersymmetric back-grounds of type II supergravity of the form $\mathbb{R}^{1,3} \times \mathcal{M}_6$ require the internal manifold $\mathcal{M}_6$ to be Calabi-Yau. Moreover all background fluxes must be set to zero, resulting in $\mathcal{N} = 2$ supersymmetry in four dimensions. Turning on the background fluxes while preserving maximal symmetry in the four non-compact dimensions forces the background to be of the form of a warped product $AdS_4 \times_w \mathcal{M}_6$, where $\mathcal{M}_6$ is no longer a Calabi-Yau.

The departure from the Calabi-Yau condition in the presence of fluxes can be elegantly described by reformulating the supersymmetry conditions in the framework of generalized geometry [1, 2]. This leads to the statement that $\mathcal{M}_6$ must possess a pair of compatible pure spinors obeying certain differential conditions [3]. However, the conditions from generalized geometry in the presence of fluxes are necessary but not sufficient, and therefore are not

quite on the same footing as the Calabi-Yau condition in the absence of fluxes: Even if one allows for the presence of supersymmetric sources, the (generalized) Bianchi identities of all form fields must be imposed in addition in order to ensure that all the equations of motion are solved [4].

Necessary and sufficient conditions for supersymmetric solutions have been established in the case of (massive) IIA backgrounds with constant dilaton and warp factor in [5]. More recently in [6] we presented what we called the 'scalar ansatz', an ansatz which solves the supersymmetry conditions of type II supergravity for backgrounds where the internal manifold possesses $SU(3) \times SU(3)$ structure. We were, however, unable to present solutions of the full set of supergravity equations of motion in the absence of sources.

In the present paper, building on the results of [6], we present a set of sufficient conditions for supersymmetric solutions of the full set of equations of motion of type IIA/IIB super-gravity in the absence of sources. The conditions can be concisely stated as follows. Let the background be of the form of a warped product $AdS_4 \times_w \mathcal{M}_6$, and let the internal manifold $\mathcal{M}_6$ be locally (but not necessarily globally) expressed as a codimension-one foliation:

$$ds^2(\mathcal{M}_6) = \mathrm{d}t^2 + ds_t^2(\mathcal{M}_5) \, , \tag{1.1}$$

where the metric of the five-dimensional leaves depends in general on the coordinate t. Let us moreover assume that on $\mathcal{M}_5$ there are three real two-forms α, β, γ and a real one-form u such that:

$$\boxed{\begin{aligned} &\iota_u \alpha = \iota_u \beta = \iota_u \gamma = 0 \\ &\alpha \wedge \beta = \beta \wedge \gamma = \gamma \wedge \alpha = 0 \\ &\alpha \wedge \alpha = \beta \wedge \beta = \gamma \wedge \gamma \neq 0 \end{aligned}} \tag{1.2}$$

and

$$\boxed{\mathrm{d}u = -2\gamma \; ; \qquad \mathrm{d}(\alpha + i\beta) = -3iu \wedge (\alpha + i\beta) \; ; \qquad \mathrm{d}\gamma = 0 \, .} \tag{1.3}$$

It then follows that $AdS_4 \times_w \mathcal{M}_6$, where the six-dimensional internal manifold is given by (1.1), is a supersymmetric pure-flux background (i.e. it does not contain any sources) of type IIA/IIB. As we show in appendix A, the conditions in eqs. (1.2,1.3) are equivalent to the statement that $\mathcal{M}_5$ admits a Sasaki-Einstein structure.

We emphasize that, provided (1.2,1.3) hold, there is no obstruction to specifying appropriate profiles for all supergravity fields so that the background $AdS_4 \times_w \mathcal{M}_6$ solves all the equations of motion of type II supergravity, not only the supersymmetry conditions. Hence eqs. (1.2,1.3) may be viewed as replacing the Calabi-Yau condition in the presence of fluxes. The precise profiles of all the fields are determined at the solution, as we explain in detail in the main text, and depend on whether one is in IIA or in IIB. In particular, all fluxes (including the Romans mass in IIA) are generally non-zero, and the dilaton and warp factor are non-constant.

The proof of the above statements relies on our results in [6]. In the present paper we construct backgrounds of the form (1.1,1.2,1.3) which as we show fulfill all the conditions

of the scalar ansatz of [6], thereby solving the supersymmetry conditions of type II supergravity. Moreover we show that all the Bianchi identities are satisfied without the need to introduce any source terms. Thanks to the integrability theorem mentioned earlier, this then implies that all the remaining equations of motion are satisfied.

The solutions presented here are expressed in terms of the real forms α, β, γ, u mentioned above, specifying a Sasaki-Einstein $SU(2)$ structure on $\mathcal{M}_5$. On the other hand, the scalar ansatz of [6] is expressed in terms of a local $SU(2)$ structure given by the triplet $(K, \omega, \widetilde{J})$, as reviewed in the main text, specifying an $SU(3) \times SU(3)$ structure on $\mathcal{M}_6$. The translation between the two descriptions is established by expressing the data of the local $SU(2)$ structure in terms of α, β, γ, u. The precise dictionary is given in eqs. (2.6,2.14) below for IIA and eq. (3.3) for IIB.

In particular the metric of $\mathcal{M}_6$ can be read off of the local $SU(2)$ structure $(K, \omega, \widetilde{J})$, as explained in [6]. The metric on the five-dimensional leaves of $\mathcal{M}_6$ picked by the supersymmetric solution is not Sasaki-Einstein, since it will not in general be the same as the metric compatible with the Sasaki-Einstein structure. As explained in appendix A, the metric of the supersymmetric background is related to the Sasaki-Einstein metric through warping and squashing; the precise relation is given in the main text. In type IIA the total six-dimensional internal space is locally an S^2 bundle over a four-dimensional Kähler-Einstein manifold, whereas in IIB it is the direct product of a circle and a five-dimensional squashed Sasaki-Einstein manifold.

Given any five-dimensional (regular or not) Sasaki-Einstein manifold (explicit examples thereof are the round S^5, the homogeneous metric on $T^{1,1}$, and the infinite $Y^{p,q}$ series [7]), we construct the corresponding families of pure-flux vacua of type II supergravity.[1] On the other hand, under the assumption of regularity, there is a correspondence between five-dimensional Sasaki-Einstein metrics and four-dimensional Kähler-Einstein manifolds of positive curvature [8]. Hence for every four-dimensional Kähler-Einstein manifold of positive curvature there is a corresponding family of vacua of type II supergravity.

Recently, the case where $\mathcal{M}_6$ is a certain circle reduction of $M^{1,1,1}$ was analyzed by Petrini and Zaffaroni in [9], and belongs to the families of vacua presented here. As in [9], the examples of section 2.3 can be viewed as 'massive deformations' of those of section 2.2, and are given in terms of a system of two coupled first-order differential equations for two unknowns. Massive deformations of general $AdS_4 \times \mathcal{M}_6$ backgrounds were recently constructed in [10] to first order in a perturbative expansion in the Romans mass.

The outline of the remainder of the paper is as follows: After a brief review of the type IIA scalar ansatz of [6] in the next section, we start with the case of $\mathcal{N} = 2$ IIA compactifications with zero Romans mass in section 2.2. This is subsequently generalized to $\mathcal{N} = 2$ massive solutions with dynamic $SU(3) \times SU(3)$ structure in section 2.3. In section 3 we review the scalar ansatz in the case of IIB and give the solution in closed form. On the IIB side we only treat the static $SU(2)$ case, although we expect the dynamic $SU(3) \times SU(3)$ case to be

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[1]Note, however, that only in the case of a regular five-dimensional Sasaki-Einstein manifold can the corresponding type IIA solution have a global extension.

a straightforward generalization thereof. The appendix A contains some useful facts about five-dimensional Sasaki-Einstein metrics, and in particular explains the relation between the metric of the five-dimensional leaves of (1.1) and the Sasaki-Einstein metric associated with the structure (1.2, 1.3). Our conclusions are contained in section 4. We have collected some useful formulæ in the appendix.

2. The IIA side

In the following subsection we start by reviewing the scalar ansatz of [6], specialized to the case of IIA. For more details the reader may consult that reference. Then in subsection 2.3 we present the $\mathcal{N} = 2$ solutions, after a brief discussion of the zero Romans mass limit in section 2.2.

2.1 Review of the scalar ansatz

The ten-dimensional spacetime metric (in the string frame) is given by:

$$ds^2 = e^{2A}ds^2(AdS_4) + ds^2(\mathcal{M}_6) \ , \tag{2.1}$$

where A is the warp factor. The internal six-dimensional manifold $\mathcal{M}_6$ is characterized by a local $SU(2)$ structure determined by the triplet $(\omega, \widetilde{J}, K)$, where ω is a complex two-form, $\widetilde{J}$ is a real two-form, and K is a complex one-form. These forms satisfy the following algebraic compatibility conditions:

$$\begin{aligned} \widetilde{J} \wedge \omega &= 0 \\ \widetilde{J} \wedge \widetilde{J} &= \operatorname{Re}\omega \wedge \operatorname{Re}\omega = \operatorname{Im}\omega \wedge \operatorname{Im}\omega \neq 0 \\ \iota_K \widetilde{J} &= \iota_K \operatorname{Re}\omega = \iota_K \operatorname{Im}\omega = 0 \ . \end{aligned} \tag{2.2}$$

Moreover, as explained in [6], associated with this local $SU(2)$ structure there are two *global $SU(3)$* structures $(J^{(i)}, \Omega^{(i)})$, $i = 1, 2$, given by:

$$\begin{aligned} J^{(1)} &= \frac{i}{2}K \wedge K^* + \widetilde{J} \ ; \qquad J^{(2)} = \frac{i}{2}K \wedge K^* - \widetilde{J} \\ \Omega^{(1)} &= -i\omega \wedge K \ ; \qquad \Omega^{(2)} = i\omega^* \wedge K \ , \end{aligned} \tag{2.3}$$

where we have normalized $|K|^2 = 2$.

The scalar ansatz introduced in [6] is an ansatz which solves the supersymmetry conditions of type II supergravity for backgrounds where the internal six-dimensional manifold possesses $SU(3) \times SU(3)$ structure. According to the ansatz one truncates the components of all the form fluxes to those which are singlets with respect to the local $SU(2)$ structure in (2.2). More specifically, the NSNS three-form is given by

$$H = \frac{1}{24}\left(h_1\omega^* + h_2\,\omega + 2h_3\widetilde{J}\right) \wedge K + \text{c.c.} \ , \tag{2.4}$$

while the RR fluxes are given by:

$$e^\phi F_0 = f_0$$
$$e^\phi F_2 = \frac{1}{8}\left(f_2\,\omega^* + f_3\widetilde{J} + 2if_1 K \wedge K^*\right) + \text{c.c.}$$
$$e^\phi F_4 = \frac{1}{16}g_1\widetilde{J} \wedge \widetilde{J} + \frac{i}{96}\left(g_2\,\omega^* + g_2^*\,\omega + 2g_3\widetilde{J}\right) \wedge K \wedge K^*$$
$$e^\phi F_6 = f\,vol_6\ , \tag{2.5}$$

where the various scalar coefficients above are given by eq. (2.15) of [6].[2] Moreover, the local $SU(2)$ structure is constrained to obey the differential conditions eqs. (2.26) of [6]. In addition one must impose the constraints in eqs. (2.16,2.17) of [6].

As explained in [6], in constructing explicit solutions the non-trivial task is to find manifolds admitting local $SU(2)$ structures such that they obey the differential conditions mentioned above. Moreover one has to worry about the Bianchi identities, which were not considered in [6]. In the following subsection we present families of solutions obeying all the conditions of the scalar ansatz, thus solving the supersymmetry equations of IIA. In fact, as we will see, the supersymmetry is $\mathcal{N} = 2$ in four dimensions (eight real supercharges). In addition we show that all the Bianchi identities are satisfied, without the need to introduce any sources. In other words the solutions correspond to pure-flux backgrounds.

2.2 Undeformed solutions

Our general solutions in the type IIA case can be viewed as families of solutions partly parameterized by the Romans mass. These families include as special cases the solutions with zero Romans mass. These special cases correspond to backgrounds which can be uplifted to solutions of eleven-dimensional supergravity and have already appeared in the literature [14]: the resulting seven-dimensional internal manifold is Sasaki-Einstein, as follows from the properties of Freund-Rubin vacua, and for a global solution it must belong to the $Y^{p,q}$ series[3].

As explained in [11], on the dual CFT_3 side the mass deformation of the supergravity background corresponds to the sum of the levels of the two Chern-Simons terms. As in [9], the deformed solutions are presented here explicitly up to a coupled system of two first-order differential equations for two unknowns.

Before coming to the general ('deformed') solutions in section 2.3, we review here the special ('undeformed') solutions with zero Romans mass. As we will see, the former correspond to

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[2]Note that there was a typo in the last line of that equation in the previous versions of ref. [6].

[3]Note however that our procedure for constructing IIA solutions can be carried out starting with any five-dimensional Sasaki-Einstein manifold, regular or not. In the latter case the SE manifold cannot be thought of as the total space of a line bundle over a globally-defined four-dimensional Kähler-Einstein manifold. The construction in our paper can nonetheless still be carried out to produce a local solution. In that case the massless, undeformed limit of the solution will not be a $Y^{p,q}$ reduction – since the latter would require a globally-defined, four-dimensional, smooth, positive-curvature Kähler-Einstein base.

backgrounds with dynamic $SU(3) \times SU(3)$ structure, while the latter possess strict $SU(3)$ structure. In both cases the solutions possess $\mathcal{N} = 2$ supersymmetry.

The supersymmetry equations and Bianchi identities for the strict $SU(3)$ structure case are summarized in appendix B. We can make use of the scalar ansatz by expressing the strict $SU(3)$ structure in terms of the local $SU(2)$ structure of $\mathcal{M}_6$. Furthermore, we will take the latter to be given by:

$$
\begin{aligned}
K &= e^A \left(\xi u + i \mathrm{d}t \right) \\
\frac{1}{3} \widetilde{J} &= e^{2A} \left(\sin\theta \, \alpha + \cos\theta \, \gamma \right) \\
\frac{1}{3} \omega &= e^{2A} \left(\cos\theta \, \alpha - \sin\theta \, \gamma + i\beta \right) \ ,
\end{aligned}
\tag{2.6}
$$

where α, β, γ are real two-forms on $\mathcal{M}_5$ and u is a real one-form on $\mathcal{M}_5$ obeying (1.2,1.3); we will assume that A, θ, ξ are all functions of t.

As explained in appendix A, it follows from the above that the metric of the six-dimensional space $\mathrm{d}s^2(\mathcal{M}_6)$ is of the form of a codimension-one foliation:

$$
ds^2(\mathcal{M}_6) = e^{2A} \left(ds_t^2(\mathcal{M}_5) + dt^2 \right) \ ,
\tag{2.7}
$$

where the metric of the five-dimensional leaves is given locally by:

$$
ds_t^2(\mathcal{M}_5) = 3ds_{KE}^2 + \xi^2 \, u \otimes u \ .
\tag{2.8}
$$

For general ξ this is a squashed Sasaki-Einstein metric. The metric is locally of the form of a $U(1)$ fibration with connection field-strength given by $\mathrm{d}u = -2\gamma$. The four dimensional base over which u is fibered is locally Kähler-Einstein with metric ds_{KE}^2.

Furthermore, we will assume that all fluxes are zero except for F_2, F_6, which are given by (B.3). Plugging the ansatz into the first of (B.5) and setting $\mathrm{Im}W = 1$ for simplicity[4], we obtain the following three equations:

$$
\begin{aligned}
\xi &= \frac{3}{2} \sin\theta \\
A' &= \frac{1}{2} \sin\theta \cos\theta \\
\theta' &= 1 + \cos^2\theta \ ,
\end{aligned}
\tag{2.9}
$$

where the prime denotes differentiation with respect to t. The first equation above determines ξ in terms of the angle θ, while the last two can be solved to determine θ and the warp factor A as functions of t:

$$
\begin{aligned}
A &= \frac{1}{4} \log \left[1 + \sin^2(\sqrt{2}\, t) \right] \\
\cos\theta &= \frac{\cos(\sqrt{2}\, t)}{\sqrt{1 + \sin^2(\sqrt{2}\, t)}} \ .
\end{aligned}
\tag{2.10}
$$

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[4]Remember that W is the inverse AdS radius and is therefore a constant.

The last line of (B.5) is automatically satisfied, by virtue of (2.9). Moreover, the second line of (B.5) can be seen to be satisfied, again taking (2.9) into account, provided:

$$\widetilde{F} = e^{-4A} \left(\frac{4}{3} - \sin^2 \theta \right) (\widetilde{J} + e^{2A} \sin \theta u \wedge \mathrm{d}t) \ . \tag{2.11}$$

As we show in appendix B, in order to have a solution to all the equations of motion we only need to make sure that the Bianchi identity (B.10) is satisfied. From (B.3,2.11), taking (2.9) into account, we can see that F_2 can be written in the form:

$$F_2 = \frac{1}{2} \mathrm{d} \left(e^{-2A} \cos \theta \, u \right) \ , \tag{2.12}$$

and therefore it satisfies the Bianchi identity (B.10).

We would like to stress that one can take $ds^2(\mathcal{M}_5)$ in (2.8) to be the squashed version of any one of the (infinitely-many) five-dimensional Sasaki-Einstein metrics, with the squashing ξ given in (2.9,2.10). The resulting $ds^2(\mathcal{M}_6)$ metric in (2.7) in particular includes as a special case the metric of the circle reduction of the $M^{1,1,1}$ manifold recently analyzed in [9]. Note however, that the case of $\mathbb{CP}^3$ is not of the form (2.7): Although $\mathbb{CP}^3$ can indeed be thought of as a codimension-one foliation with leaves given by $T^{1,1}$, the metric is not of the form (2.8).

Moreover one can show that the foliations (2.7) are smooth. To see this note that potential singularities arise at the zeros of $\xi(t)$, which occur at $t_0 = n\pi/\sqrt{2}$, $n \in \mathbb{Z}$. Let us set $u = d\psi + \mathcal{A}$, where ψ is the coordinate of the $U(1)$ fiber and $\mathcal{A}$, such that $\mathrm{d}\mathcal{A} = -2\gamma$, is the $U(1)$ connection. Near t_0 the metric takes the form:

$$ds^2(\mathcal{M}_6) \approx e^{2A(t_0)} \left(dt^2 + 9(t - t_0)^2 (d\psi + \mathcal{A})^2 + 3ds^2_{KE} \right) \ , \tag{2.13}$$

where we have taken (2.9,2.10) into account. Assuming the warp factor does not blow up at t_0, this will be free of singularities provided we take ψ to have period $2\pi/3$.

The above argument also shows that if we take t to have the range $0 \leq t \leq \pi/\sqrt{2}$, (2.7) can be thought of locally as an S^2 fibration over the four-dimensional Kähler-Einstein base. Indeed, fixing the point on the four-dimensional base, the (ψ, t) fiber is a circle parametrized by ψ which is fibered over t. Moreover the circle smoothly shrinks to zero size at the endpoints of the interval $t = 0, \pi/\sqrt{2}$, showing that the (ψ, t) fiber has the topology of S^2.

Finally let us remark that the solutions presented here possess $\mathcal{N} = 2$ supersymmetry. Indeed this is a consequence of the fact that, thanks to (2.12), the fluxes do not depend on the two-forms α, β. Moreover, as explained in appendix A, the metric is invariant under general orthogonal rotations of the triplet α, β, γ. Consequently all fields are invariant under $SO(2)$ rotations in the (α, β) plane. (Note that these rotations would have to be t-independent for them to leave eqs. (1.3) invariant.)

2.3 Mass-deformed solutions

We are now ready to generalize the solutions of the previous subsection to include non-zero Romans mass and dynamic $SU(3) \times SU(3)$ structure.

We will take the local $SU(2)$ structure of $\mathcal{M}_6$ to be given by:

$$
\begin{aligned}
K &= e^B \left(\mathrm{d}t - i\xi u \right) \\
\frac{W}{3} \widetilde{J} &= e^{2C} \left\{ \sin\theta \left(\cos\zeta\,\alpha - \sin\zeta\,\beta \right) + \cos\theta\,\gamma \right\} \\
\frac{W}{3} \omega &= e^{2C} \left\{ \cos\theta \left(\cos\zeta\,\alpha - \sin\zeta\,\beta \right) - \sin\theta\,\gamma + i(\cos\zeta\,\beta + \sin\zeta\,\alpha) \right\} ,
\end{aligned}
\tag{2.14}
$$

where we have set $W \in \mathbb{R}$. As before α, β, γ are real two-forms on $\mathcal{M}_5$ and u is a real one-form on $\mathcal{M}_5$ obeying (1.2,1.3); we take B, C, θ, ζ to be functions of t. Note that, up to the different warp factors, the deformed ansatz above is obtained from the undeformed one in (2.6) via a t-dependent $SO(2)$ rotation in the (α, β) plane through angle $\zeta(t)$.

Furthermore we take the spinor ansatz corresponding to the local $SU(2)$ structure above (*cf.* eq. (2.7) of [6]) to be given by:

$$
\theta_1 = e^{\frac{1}{2}A}\eta_1 ; \qquad \theta_2 = -e^{\frac{1}{2}A} \left(\sin\varphi\,\eta_2^* + ie^{i\varepsilon} \cos\varphi\,\eta_1^* \right) ,
\tag{2.15}
$$

with φ, ε functions of t. These angles will turn out to be non-constant, thus corresponding to a dynamic $SU(3) \times SU(3)$ structure. As before, the warp factor A is taken to be a function of t. In addition the angle θ in (2.14) obeys:

$$
\tan\theta = \frac{\tan\varphi}{\sin\varepsilon} .
\tag{2.16}
$$

It follows from the above that the metric of the six-dimensional space $\mathrm{d}s^2(\mathcal{M}_6)$ is locally of the form of a codimension-one foliation:

$$
ds^2(\mathcal{M}_6) = e^{2B} \left(\frac{3}{W} e^{2(C-B)} ds_{KE}^2 + \xi^2\, u \otimes u + dt^2 \right) .
\tag{2.17}
$$

Note that we could use up the reparameterization invariance of t to set either one of B or ξ to some given function of t. This redundancy will prove useful in the following. Furthermore the fluxes are given by (2.5), where:

$$
\begin{aligned}
f &= -3We^{-A} \cos\varepsilon \cos\varphi \\
f_0 &= -We^{-A} \left(\cos\varphi \sin\varepsilon + \csc\varepsilon \sin\varphi \tan\varphi \right) \\
f_1 &= -\cos\varepsilon \cos\varphi \left(We^{-A} + 4e^{-B} A' \cot\varphi \sin\varepsilon \right) \\
f_2 &= -8e^{-B} A' \cos\varepsilon \cos\varphi \\
g_1 &= -8 \left(\cos\varphi \sin\varepsilon + \sin\varphi \csc\varepsilon \tan\varphi \right) \left(We^{-A} - 4e^{-B} A' \cot\varphi \sin\varepsilon \right) \\
g_2 &= 48 \sin\varphi \left(We^{-A} + e^{-B} A' \cot\varphi \sin\varepsilon \right) \\
h_1 &= -6 \sin^2\varphi \cot\varepsilon \left(We^{-A} - 2e^{-B} A' \sin\varepsilon \cot\varphi \right) \\
\frac{h_1}{h_3} &= \frac{h_2}{h_3} = \frac{f_2}{f_3} = \frac{g_2}{g_3} = -\tan\theta .
\end{aligned}
\tag{2.18}
$$

With the above equations, it is straightforward to verify that all conditions of the scalar ansatz of [6] are satisfied, provided the following equations hold:

$$e^{4A} = \frac{1}{\cos^2\theta}\tan\varepsilon$$

$$e^{B-A} = -\frac{1}{2W}\cot\theta\,(\log\tan\varepsilon)'$$

$$e^{\phi-3A} = \cos\varphi\cos\varepsilon$$

$$\xi = \frac{3}{2W}e^{A-B}\sin\theta \tag{2.19}$$

$$\zeta' = \frac{1}{2W}e^{2(A-C)}\cos\theta\cot\varepsilon\sin^2\varphi\,(\log\tan\varepsilon)'$$

$$\theta' = \cot\theta\left(\frac{1}{2W}e^{2(A-C)}\sin^2\theta - 1\right)(\log\tan\varepsilon)'$$

$$C' = -\frac{1}{4W}e^{2(A-C)}(\sin^2\varphi + \cos^2\theta)(\log\tan\varepsilon)'\ .$$

Taking (2.16) into account and using a t-coordinate transformation to fix θ to some given function of t, it readily follows that the first five of the system of equations (2.19) solve for A, B, ϕ, ξ, ζ in terms of C, ε. Moreover, the last two equations in (2.19) is a system of two coupled first-order differential equations for the two unknowns C, ε. This is exactly as in [9]. Unfortunately we will not be able to provide an analytical solution for this system here, but will note that it can be analyzed perturbatively using numerical methods [9].

It is a tedious but straightforward calculation to show that all the Bianchi identities are satisfied without further constraints. To somewhat simplify the computation one may choose the 'gauge' $B = A$ in order to fix the redundancy in the definition of the coordinate t. It is also useful to take the formulæ in appendix D into account.

We can use the same argument as in the undeformed case to show that the foliations (2.17) are smooth, provided the period of the coordinate of the $U(1)$ fiber is chosen appropriately, and that the six-dimensional metric can also be thought of locally as an S^2 fibration over the four-dimensional Kähler-Einstein base. For example, using a t-coordinate transformation to fix θ to be the same as in the undeformed case and assuming the warp factors do not blow up, the discussion around (2.13) carries over virtually unchanged.

Finally let us remark that, as in the undeformed case, the solutions presented here possess $\mathcal{N} = 2$ supersymmetry. This follows from the fact that, thanks to (2.18), the fluxes do not depend on the two-forms α, β. Consequently all fields are invariant under t-independent $SO(2)$ rotations in the (α, β) plane, and hence there is an $SO(2)$-worth of $SU(3) \times SU(3)$ structures satisfying the supersymmetry conditions.

3. The type IIB side

Let us start by reviewing the scalar ansatz of [6] for type IIB solutions, specializing to the case of static $SU(2)$. The ten-dimensional spacetime metric is again of the form (2.1). The

NSNS three-form is given by

$$H = \frac{1}{24}\left(h_1\omega^* + h_2\,\omega + 2h_3\widetilde{J}\right) \wedge K + \text{c.c.} \;, \tag{3.1}$$

while the RR fluxes are given by:

$$e^\phi F_1 = g_1 K + \text{c.c.}$$

$$e^\phi F_3 = \frac{1}{24}\left(f_1\omega^* + f_2\,\omega + 2f_3\widetilde{J}\right) \wedge K + \text{c.c.} \tag{3.2}$$

$$e^\phi F_5 = g_2 \star_6 K + \text{c.c.} \;,$$

where the various scalar coefficients above are given by eq. (4.1) of [6]. Moreover, the static $SU(2)$ structure is constrained to obey the differential conditions eqs. (4.3) of [6]. In addition one must impose the constraints in eqs. (4.2) of that reference.

We take the local $SU(2)$ structure to be given by:

$$K = e^A\left(\frac{6}{5W}u + i\mathrm{d}t\right)$$

$$\frac{5W^2}{6}\widetilde{J} = e^{2A}\left(\sin\theta\,\alpha + \cos\theta\,\beta\right) \tag{3.3}$$

$$\frac{5W^2}{6}\omega = e^{2A}\left(\cos\theta\,\alpha - \sin\theta\,\beta - i\gamma\right) \;,$$

where α, β, γ are real two-forms on $\mathcal{M}_5$ and u is a real one-form on $\mathcal{M}_5$ obeying (1.2,1.3). The corresponding six-dimensional metric reads:

$$ds^2(\mathcal{M}_6) = e^{2A(t)}\left(ds^2(\mathcal{M}_5) + dt^2\right) \;, \tag{3.4}$$

where

$$ds^2(\mathcal{M}_5) = \frac{6}{5W^2}\left(ds^2_{KE} + \frac{6}{5}u\otimes u\right) \;, \tag{3.5}$$

and we have taken $W \in \mathbb{R}$. This is the local form of the metric of a squashed five-dimensional Sasaki-Einstein manifold.

The NSNS three-form is given by

$$H = \frac{1}{2}W\mathrm{Re}\omega \wedge \mathrm{d}t - \left(2A'\widetilde{J} + ce^{-4A}\mathrm{Re}\omega\right) \wedge e^{-A}\mathrm{Re}K \;, \tag{3.6}$$

where c is a real constant. The RR fluxes are given by:

$$e^\phi F_1 = -2ce^{-4A}\mathrm{d}t$$

$$e^\phi F_3 = -\frac{1}{2}W\widetilde{J}\wedge \mathrm{d}t + \left(2A'\mathrm{Re}\omega - ce^{-4A}\widetilde{J}\right) \wedge e^{-A}\mathrm{Re}K \tag{3.7}$$

$$e^\phi F_5 = \frac{3}{2}W\widetilde{J}\wedge \widetilde{J}\wedge e^{-A}\mathrm{Re}K \;,$$

while the dilaton is related to the warp factor through:

$$\phi = 4A \;. \tag{3.8}$$

It is now straightforward to verify that all the Bianchi identities are satisfied, provided we take:

$$e^{4A} = \begin{cases} \frac{2}{\sqrt{5}} \left| \frac{c}{W} \right| \cosh \left[\sqrt{5} W (t - t_0) \right] , & c \neq 0 \\ \exp \left[\sqrt{5} W (t - t_0) \right] , & c = 0 \end{cases} , \tag{3.9}$$

and:

$$\theta = \begin{cases} \arctan \tanh \left[\frac{\sqrt{5}}{2} W (t - t_0) \right] + \theta_0 , & c \neq 0 \\ \theta_0 , & c = 0 \end{cases} , \tag{3.10}$$

for some constant θ_0. The real constant c distinguishing the two different cases above is the same one as in eqs. (3.6,3.7). It follows that in the absence of F_1 flux ($c = 0$) the solution is a linear dilaton background.

The ten-dimensional metric in the Einstein frame is of direct-product form:

$$ds_E^2 = ds^2(AdS_4) + ds^2(\mathcal{M}_5) + dt^2 , \tag{3.11}$$

as follows from (3.8). However, we suspect that this feature is an artifact of the static $SU(2)$ structure of the solution. We do not expect more general $SU(3) \times SU(3)$-structure solutions to be of direct-product form.

We may choose to compactify the t-direction by a coordinate transformation. For example, considering the $c \neq 0$ case we can take:

$$\sqrt{5} W (t - t_0) = \log \tan \frac{\chi}{4} , \tag{3.12}$$

upon which the dilaton takes the form:

$$e^\phi = \frac{2}{\sqrt{5}} \left| \frac{c}{W} \right| \frac{1}{\sin \frac{\chi}{2}} . \tag{3.13}$$

Hence in the compactified description the solution is singular.

4. Conclusions

We have presented a set of sufficient conditions for the existence of supersymmetric backgrounds of IIA/IIB supergravity of the form $AdS_4 \times_w \mathcal{M}_6$. The conditions state that the internal six-dimensional manifold should be locally (but not necessarily globally) a codimension-one foliation, such that the five-dimensional leaves admit a Sasaki-Einstein structure. In type IIA the supersymmetry is $\mathcal{N} = 2$, and the total six-dimensional internal space is locally an S^2 bundle over a four-dimensional Kähler-Einstein manifold; in IIB the internal space is the direct product of a circle and a five-dimensional squashed Sasaki-Einstein manifold.

The solutions presented here are of obvious relevance to the AdS_4/CFT_3 correspondence. Recently, the case where $\mathcal{M}_6$ is a certain circle reduction of $M^{1,1,1}$ was analyzed by Petrini and Zaffaroni in [9]. As in [9], the examples of section 2.3 can be viewed as 'massive

deformations' of those of section 2.2, and are given in terms of a system of two coupled first-order differential equations for two unknowns.

Massive deformations of general $AdS_4 \times \mathcal{M}_6$ backgrounds, including the $AdS_4 \times \mathbb{CP}^3$ background [13] as a special case, were recently constructed in [10] to first order in a perturbative expansion in the Romans mass. Both $\mathbb{CP}^3$ and the circle reduction of $M^{1,1,1}$ considered in [9] can be viewed as codimension-one foliations with five-dimensional leaves admitting Sasaki-Einstein structures. However, as explained in section 2.2, only in the case of $M^{1,1,1}$ is the foliation of the precise form considered here; the $AdS_4 \times \mathbb{CP}^3$ type IIA solution is not among those of section 2.2. Instead we have a foliation with $T^{1,1}$ leaves such that the total space is locally an S^2 bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Given any five-dimensional Sasaki-Einstein manifold (regular or not), we have constructed corresponding families of pure-flux vacua of type II supergravity with all fluxes (including the Romans mass in IIA) generally non-zero. Explicit examples of five-dimensional Sasaki-Einstein spaces are the round S^5, the homogeneous metric on $T^{1,1}$, and the infinite $Y^{p,q}$ series [7].[5] On the other hand, under the assumption of regularity, there is a correspondence between five-dimensional Sasaki-Einstein metrics and four-dimensional Kähler-Einstein manifolds of positive curvature [8]. Hence for every four-dimensional Kähler-Einstein manifold of positive curvature we have constructed corresponding families of vacua of type II supergravity.

The massless IIA vacua presented in section 2.2 can be uplifted to Freund-Rubin vacua of eleven-dimensional supergravity. The resulting seven-dimensional internal manifold must be Sasaki-Einstein, as follows from the properties of Freund-Rubin vacua. Recall that in type IIA the internal six-dimensional manifold $\mathcal{M}_6$ can be viewed locally as an S^2 bundle over a four-dimensional Kähler-Einstein base. Hence, given any four-dimensional Kähler-Einstein manifold of positive curvature there is a corresponding seven-dimensional Sasaki-Einstein one which is a circle fibration over $\mathcal{M}_6$. Indeed, this should precisely correspond to the construction of [14]. This can also be seen explicitly from the type IIA reduction of the solutions discussed in [15], *cf.* section 5.1 therein.[6] In other words, the type IIA solutions presented here include the massive deformations of the IIA reduction of the $Y^{p,q}$ solutions discussed in [15].

The solutions presented here are by no means the most general. Even within the framework of the scalar ansatz, it would be interesting to try to extend our solutions by e.g. generalizing the dependence of the local $SU(2)$ structure on the Euler angles in eqs. (2.14). As already remarked, in the case of IIB our solutions are of the static $SU(2)$ type. However, we expect the generalization to dynamic $SU(3) \times SU(3)$ structure to be straightforward. We expect that for such generalized backgrounds the internal six-dimensional manifold would no longer possess a direct-product structure. We hope to report on this in the near future. The IIB backgrounds presented in section 3 could be Wick-rotated to obtain cosmological

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[5]Note, however, that only in the case of a regular five-dimensional Sasaki-Einstein manifold can the corresponding type IIA solution have a global extension.

[6]We thank D. Martelli for bringing this to our attention.

solutions with time-dependent dilaton. These may be amenable to analysis with conformal field theory techniques. It would be interesting to pursue this further.

The results of the present paper, which relied on the 'scalar ansatz' of [6], suggest that four-dimensional Kähler-Einstein manifolds play a central role in flux compactifications. Smooth four-dimensional Kähler-Einstein manifolds of positive curvature were classified in [16]: they are $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, and the del Pezzo surfaces $dP_3, \ldots, dP_8$. It is intriguing that the latter have been shown to play a special role in recent F-theoretic constructions with phenomenological applications [17]. It remains to be seen whether this mathematical structure persists beyond the present setup.

A. Five-dimensional Sasaki-Einstein manifolds

In this section we show that eqs. (1.2,1.3) are equivalent to the statement that $\mathcal{M}_5$ admits a Sasaki-Einstein structure. Moreover we spell out the precise relation between the Sasaki-Einstein metric and the metric picked by the supersymmetric background. As we will see, the latter is obtained from the former by warping and squashing.

In five dimensions a Sasaki-Einstein manifold may be defined under certain additional mild assumptions (see for example [8], or theorem 5.1.6 of [12]) as one which admits a pair of Killing spinors, related by complex conjugation, obeying:

$$\nabla_m \eta = \pm \frac{i}{2} \Gamma_m \eta \ . \tag{A.1}$$

Let us assume that η_1 is a Killing spinor obeying (A.1) with the positive sign, and let us define

$$\eta_2 := C \eta_1^* \ . \tag{A.2}$$

It then follows that η_2 obeys (A.1) with the negative sign. (For our spinor conventions the reader may consult section C). With the above normalization the metric is Einstein so that the Ricci tensor of $\mathcal{M}_5$ is given by:

$$R_{mn} = 4g_{mn} \ , \tag{A.3}$$

and therefore the six-dimensional cone $\mathcal{C}(\mathcal{M}_5)$ is Calabi-Yau.

It follows from (A.1) that the norm of η_1 (which is equal to the norm of η_2) is constant. We will take the norm of $\eta_{1,2}$ to be given by:

$$\eta_1^\dagger \eta_1 = \eta_2^\dagger \eta_2 = 1 \ . \tag{A.4}$$

Moreover let us define a real one-form u:

$$u_m := (\widetilde{\eta}_2 \Gamma_m \eta_1) \ , \tag{A.5}$$

where we used definition (C.4), and three real two-forms α, β, γ:

$$\begin{aligned} \alpha_{mn} + i\beta_{mn} &:= (\widetilde{\eta}_1 \Gamma_{mn} \eta_1) = (\widetilde{\eta}_2 \Gamma_{mn} \eta_2)^* \\ \gamma_{mn} &:= i(\widetilde{\eta}_2 \Gamma_{mn} \eta_1) \ . \end{aligned} \tag{A.6}$$

A little bit of Fierzing reveals that the forms defined above satisfy all the algebraic conditions (1.2). To see this, it is useful to note that (A.4,A.5,A.6,C.3) imply:

$$\eta_1 \widetilde{\eta}_2 = \frac{1}{4} \left\{ -1 + u_m \Gamma^m + \frac{i}{2} \gamma_{mn} \Gamma^{mn} \right\} \ . \tag{A.7}$$

Similarly, taking the Killing spinor equation into account one can show that all the differential equations in (1.3) are satisfied.

We have thus showed that (1.2,1.3) follow from the assumption that $\mathcal{M}_5$ admits a Sasaki-Einstein metric. The converse can also be seen as follows: Let $\mathcal{M}_5$ be an $SU(2)$-structure manifold. It follows that on $\mathcal{M}_5$ there is a globally-defined nowhere-vanishing spinor η_1 with associated $SU(2)$ structure (1.2). If $\mathcal{M}_5$ does not admit a Sasaki-Einstein metric $\nabla_m \eta_1$ would be given by the right-hand side of (A.1) plus additional terms. It can then be seen, by similar manipulations as above, that these additional terms would violate (some of) the equations in (1.3).

The Sasaki-Einstein metric associated with the $SU(2)$ structure (1.2,1.3) can locally be put in the canonical form:

$$ds^2_{SE} = ds^2_{KE} + u \otimes u \ , \tag{A.8}$$

where ds^2_{KE} is a Kähler-Einstein four-dimensional base over which u is fibered. The connection field strength of this local $U(1)$ fibration is the Kähler form of the base, and is equal to $du = -2\gamma$. If in addition the orbits of the vector[7] dual to u close and the associated $U(1)$ action is free, (A.8) extends globally and the base is a four-dimensional Kähler-Einstein manifold of positive curvature.

The $SU(2)$ structure (1.2) possesses an $SU(2)$ invariance which also leaves the associated metric (A.8) invariant. In order to see this, let us define the triplet of real three forms:

$$\vec{J} := -\frac{i}{2} \vec{\sigma}_{ij} (\eta_i^\dagger \Gamma_{(2)} \eta_j) \ , \tag{A.9}$$

where $\vec{\sigma}$ is a triplet of Pauli matrices, so that $\alpha = J^{(2)}$, $\beta = J^{(1)}$, $\gamma = J^{(3)}$. Up to normalization these obey:

$$J^{(a)} \wedge J^{(b)} = \delta^{ab} \text{vol}_4 \ , \quad a, b = 1, 2, 3 \ , \tag{A.10}$$

where vol_4 is the volume element of the four-dimensional base of the fibration (A.8). It is a straightforward computation to show that under infinitesimal $SU(2)$ transformations of the spinors $\eta_{1,2}$,

$$\delta \eta_i = \frac{i}{2} \delta \vec{\theta} \cdot \vec{\sigma}_{ij} \eta_j \ , \tag{A.11}$$

the forms $J^{(a)}$ transform as a vector of $SO(3)$:

$$\delta \vec{J} = -\delta \vec{\theta} \times \vec{J} \ . \tag{A.12}$$

Both transformations (A.11,A.12) leave the associated metric invariant.

---

[7]This is known as the Reeb vector; as follows from (A.1,A.5), it is Killing and has unit norm.

One can also see the $SU(2)$ invariance of the metric directly as follows: Choosing the orthonormal frame so that:

$$J^{(1)} + iJ^{(2)} = e_1 \wedge e_2 \ ; \qquad J^{(3)} = -\frac{i}{2}(e_1 \wedge e_1^* + e_2 \wedge e_2^*) \ , \tag{A.13}$$

the metric (A.8) can be written as:

$$ds_{SE}^2 = e_1 \otimes e_1^* + e_2 \otimes e_2^* + u \otimes u \ . \tag{A.14}$$

It is then straightforward to read off the action of (A.12) on the e_i's and show that it leaves the metric invariant. For example, infinitesimal rotations of the form (A.12) in the $(1,2)$ plane imply $\delta e_{1,2} = \frac{i}{2}\delta\theta\, e_{1,2}$, and similarly for the $(1,3)$ and $(2,3)$ planes.

We can now state the precise relation between the metric of $\mathcal{M}_6$ associated with the triplet $(K, \widetilde{J}, \omega)$ in (2.6) and the metric associated with the Sasaki-Einstein structure (1.2). From the discussion above and the fact that triplet $\mathrm{Re}\omega$, $\mathrm{Im}\omega$, $\widetilde{J}$ is obtained up to rescalings from the triplet α, β, γ by an $SO(3)$ rotation, it follows that the metric on $\mathcal{M}_6$ is given by:

$$ds^2(\mathcal{M}_6) = e^{2A}\left(3ds_{KE}^2 + \xi^2(t)\,u \otimes u\right) + e^{2A}\,\mathrm{d}t \otimes \mathrm{d}t \ , \tag{A.15}$$

where ds_{KE}^2 is the four dimensional Kähler-Einstein metric of eq. (A.8). This is a codimension-one foliation with five-dimensional leaves $\mathcal{M}_5$. As advertised, the metric $ds^2(\mathcal{M}_5)$ is obtained from the Sasaki-Einstein metric (A.8) by a warping given by $e^{2A(t)}$ and a squashing given by $\xi(t)/\sqrt{3}$. The metrics in (2.17), (3.4) are obtained by the same reasoning.

B. Massless IIA with strict $SU(3)$ structure

For non-zero Romans mass, the case of strict $SU(3)$ structure was considered in [5]. As was shown in that reference, the dilaton and warp factor must then be constant. In the case of zero Romans mass it is possible to generalize this to include non-constant warp factor and dilaton, provided:

$$\phi = 3A \ , \tag{B.1}$$

as can be seen from e.g. eq. (2.16) of [6]. Moreover, in the conventions of [6] which we follow here, taking the zero Romans mass limit requires setting the real part of the inverse AdS_4 radius to zero:

$$W = i\mathrm{Im}W \ . \tag{B.2}$$

The only non-zero fluxes are given by:

$$e^\phi F_2 = -2\mathrm{d}A\lrcorner\mathrm{Re}\Omega - \frac{1}{3}e^{-A}\mathrm{Im}W J + e^\phi\widetilde{F}$$
$$e^\phi F_6 = -3e^{-A}\mathrm{Im}W\, vol_6 \ , \tag{B.3}$$

where $\widetilde{F}_2$ is a primitive piece. The non-zero $SU(3)$ torsion classes are given by:

$$\mathcal{W}_1 = -\frac{4i}{3}e^{-A}\mathrm{Im}W$$
$$\mathcal{W}_2 = ie^\phi\widetilde{F} \tag{B.4}$$
$$\mathcal{W}_5 = (\mathrm{d}A)^{1,0} \ ,$$

so that:

$$dJ = 2e^{-A}\mathrm{Im}W\mathrm{Re}\Omega$$

$$d\left(e^{-A}\mathrm{Im}\Omega\right) = -\frac{4}{3}e^{-2A}\mathrm{Im}WJ \wedge J + e^{2A}\widetilde{F} \wedge J \tag{B.5}$$

$$d\left(e^{-A}\mathrm{Re}\Omega\right) = 0 .$$

Requiring $d^2 = 0$ on (J, Ω) is equivalent to:

$$J \wedge d(e^{2A}\widetilde{F} - \frac{4}{3}e^{-2A}\mathrm{Im}WJ) = 0 . \tag{B.6}$$

The Bianchi identities for the RR six-form:

$$dF_6 = d\left(e^{4A} \star_6 F_6\right) = 0 , \tag{B.7}$$

can be seen to be automatically satisfied. Moreover, the Bianchi identity for the RR two-form:

$$d\left(e^{4A} \star_6 F_2\right) , \tag{B.8}$$

is also automatically satisfied, as can be seen by taking (B.5) into account and using:

$$\star_6 \left(dA \lrcorner \mathrm{Re}\Omega\right) = -dA \wedge \mathrm{Im}\Omega . \tag{B.9}$$

The identity above can be derived by expressing Ω in terms of the local $SU(2)$ structure, $\Omega = -i\omega \wedge K$, where $(dA)^{1,0} = \frac{1}{2}K(K^*\partial A)$. The remaining Bianchi identity:

$$dF_2 = 0 , \tag{B.10}$$

does not follow automatically and therefore imposes an additional constraint.

C. Spinor conventions in five dimensions

In this section we list our spinor conventions in five Euclidean dimensions, which are used in appendix A.

The irreducible spinor representation is four-dimensional pseudoreal. The charge conjugation and the gamma matrices obey:

$$C^{Tr} = -C ; \qquad (\Gamma_m C)^{Tr} = -\Gamma_m C . \tag{C.1}$$

The five-dimensional Hodge operator acts on the gamma matrices as follows:

$$\star\Gamma^{(5-k)} = (-)^{\frac{1}{2}k(k-1)}\Gamma^{(k)} , \tag{C.2}$$

where $\Gamma^{(k)}$ is the antisymmetrized product of k gamma matrices.

The Fierz identity reads:

$$\chi\widetilde{\psi} = -\frac{1}{4}\left\{(\widetilde{\psi}\chi) + (\widetilde{\psi}\Gamma_m\chi)\Gamma^m + \frac{1}{2}(\widetilde{\psi}\Gamma_{mn}\chi)\Gamma^{mn}\right\} , \tag{C.3}$$

for any pair of commuting spinors χ, ψ, where we have defined:

$$\widetilde{\psi} := \psi^{Tr}C^{-1} . \tag{C.4}$$

D. Useful identities

The following identities are useful in verifying the Bianchi identities of the various solutions presented in the main text:

$$
\begin{aligned}
\star(\widetilde{J} \wedge \widetilde{J} \wedge K) &= -2iK \\
\star(\omega \wedge K) &= -i\omega \wedge K \\
\star(\omega^* \wedge K) &= -i\omega^* \wedge K \\
\star(\widetilde{J} \wedge K) &= -i\widetilde{J} \wedge K \\
\star J &= -\frac{1}{2} J \wedge J \\
\star\omega &= -\frac{i}{2} \omega \wedge K \wedge K^* \\
\star K &= -\frac{i}{2} \wedge \widetilde{J} \wedge \widetilde{J} \wedge K \ ,
\end{aligned}
\tag{D.1}
$$

where the Hodge star above is with respect to the internal six-dimensional metric.

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